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COEFFICIENT BOUNDS AND FEKETE–SZEGÖ INEQUALITIES FOR BI-UNIVALENT FUNCTIONS ASSOCIATED WITH BALANCING POLYNOMIALS

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Abstract

In this paper, we introduce and investigate a novel subclass of bi-univalent functions associated with balancing polynomials. By employing the techniques of (p,q)-calculus and polynomial subordination, we derive coefficient bounds for the initial coefficients a_2, a_3, and a_4 of functions in this class. Furthermore, we establish Fekete–Szegö inequalities for the introduced class. The results obtained generalize and extend various well-known results in the existing literature.

1. Introduction

The theory of bi-univalent functions has gained considerable attention in recent years, primarily due to its applications in complex analysis and geometric function theory. Let Σ denote the class of functions f analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0$, $f'(0) = 1$, and possessing a Taylor–Maclaurin series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

A function $f \in \Sigma$ is said to be bi-univalent if both f and its inverse f^{-1} are univalent in Δ. The class Σ of bi-univalent functions was first introduced and studied by Lewin (1967), who obtained the bound $|a_2| < 1.51$. Brannan and Clunie (1980) later posed the problem of finding nontrivial bounds for the coefficients a_2, a_3 for functions in Σ. Since then, numerous subclasses of Σ have been investigated in order to obtain coefficient estimates, Fekete–Szegö inequalities, and related results

2.Preliminaries:

We briefly recall the concepts of (p,q)-calculus, balancing polynomials, and then introduce the subclass under consideration.

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Definition 2.1. The (p,q)-integer $[n]_{(p,s)}$ is defined by:

form:

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \text{ where}$$

The sequence of balancing polynomials $\{B_n(x)\}$ is defined recursively by:

$$B_0(x) = 0, B_1(x) = 1, B_{n+1}(x) = 6x\, B_n(x) - B_{n-1}(x)$$

Motivated by these notions, we define the following subclass of Σ:

Definition 2.2. A function f ∈ Σ belongs to the class $\Sigma_B(\mu, p, q)$ if the subordination relation

$$1 + \frac{1}{\mu}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \Psi(z)$$

and its inverse $g = f^{-1}$ satisfy the analogous condition, where Ψ(z) is the generating function associated with balancing polynomials given by:

$$\Psi(z) = \sum_{n=0}^{\infty} B_n(x) z^n$$

3. Coefficient Bounds

Theorem 3.1. Let f ∈ $\Sigma_B(\mu, p, q)$be given by $f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$. Then the following coefficient bounds hold:

$$|a_2| \leq \sqrt{\frac{|\Psi_2|}{|\Psi_3 - \Psi_2^2|}}$$

$$|a_3| \leq \frac{|\Psi_2| + |\Psi_3|}{|\Psi_3 - \Psi_2^2|}$$

Let f ∈ $\Sigma_B(\mu, p, q)$then, there exist Schwarz functions \$ w(z) \$ and \$ v(w) \$ such that:

$$(1 - \lambda)\big(\mathcal{J}_{p,q} f\big)(z) + \lambda\big(D_{p,q}\mathcal{J}_{p,q} f\big)(z) = \mathcal{G}(s, t; w(z))$$

$$(1 - \lambda)\big(\mathcal{J}_{p,q} g\big)(w) + \lambda\big(D_{p,q}\mathcal{J}_{p,q} g\big)(w) = \mathcal{G}(s, t; v(w))$$

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where g(w) is the inverse of f(z) .

By construction, the left-hand side expands as:

$$z+\Psi_2 a_2 z^2+\Psi_3 a_3 z^3+\Psi_4 a_4 z^4+\cdots$$

With $$\Psi_n=\frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n+\lambda+1)}\left(1+\lambda[n]_{p,q}\right)$$

The right-hand side, using subordination, can be expressed as:

$$\mathcal{G}(s,t;w(z))=1+B_1(s,t)w(z)+B_2(s,t)w(z)^2+B_3(s,t)w(z)^3+\cdots$$

Since $$\mathcal{G}(s,t;z)\prec\frac{1+\tau z}{1-\mu z}$$

there exists a Schwarz function $\varphi(z)$ such that:

$$\mathcal{G}(s,t;z)=\frac{1+\tau\varphi(z)}{1-\mu\varphi(z)}=1+(\tau+\mu)\varphi(z)+(\tau+\mu)\mu\varphi(z)^2+(\tau+\mu)\mu^2\varphi(z)^3+\cdots$$

Let $w(z)=c_1z+c_2z^2+c_3z^3+\cdots$

$$\mathcal{G}(s,t;w(z))=1+B_1(s,t)c_1z+[B_1(s,t)c_2+B_2(s,t)c_1^2]z^2+[B_1(s,t)c_3+2B_2(s,t)c_1c_2+B_3(s,t)c_1^3]z^3+\cdots$$

The alternative expansion yields:

$$\mathcal{G}(s,t;w(z))=1+P_1c_1z+[P_1c_2+P_2c_1^2]z^2+[P_1c_3+2P_2c_1c_2+P_3c_1^3]z^3+\cdots$$

Where $$P_n=(\tau+\mu)\mu^{n-1}$$

$$\Psi_2a_2=P_1c_1 \qquad (1)$$

$$\Psi_4a_4=P_1c_3+2P_2c_1c_2+P_3c_1^3 \qquad (3)$$

For the inverse function, let

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3+\cdots$$

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$$(1-\lambda)\big(\mathcal{J}_{p,q}g\big)(w)+\lambda\big(D_{p,q}\mathcal{J}_{p,q}g\big)(w)$$
$$= w+\Psi_2(-a_2)w^2+\Psi_3(2a_2^2-a_3)w^3+\Psi_4(-5a_2^3+5a_2a_3$$
$$-a_4)w^4+\cdots$$

Let $$\mathrm{d}(w)=d_1w+d_2w^2+d_3w^3+\cdots$$

Then,

$$\mathcal{G}(s,t;v(w))=1+P_1d_1w+[P_1d_2+P_2d_1^2]w^2+[P_1d_3+2P_2d_1d_2+P_3d_1^3]w^3+\cdots$$

From coefficient comparison:

$$-\Psi_2a_2=P_1d_1 \tag{4}$$

$$\Psi_3(2a_2^2-a_3)=P_1d_2+P_2d_1^2 \tag{5}$$

$$-\Psi_4(5a_2^3-5a_2a_3+a_4)=P_1d_3+2P_2d_1d_2+P_3d_1^3 \tag{6}$$

From (1) and (4):

$$\Psi_2a_2=P_1c_1$$

$$-\Psi_2a_2=P_1d_1 \implies c_1=-d_1$$

$$|\Psi_2|^2|a_2|^2=|P_1|^2|c_1|^2\le|P_1|^2$$

|c1|≤1|,

$$|a_2|\le\frac{|P_1|}{|\Psi_2|}=\frac{|\tau+\mu|}{|\Psi_2|}$$

:

$$|a_2|\le\frac{|\tau-\mu|}{|\Psi_2|}$$

From previous coefficient relations:

$$2a_2^2\left(\Psi_3-\frac{P_2\Psi_2^2}{P_1^2}\right)=P_1(c_2+d_2)$$

$$|a_2|^2 \leq \frac{|P_1|}{|\Psi_3 - \frac{P_2\Psi_2^2}{P_1^2}|}$$

Substitute $P_2 = (\mu / (\tau + \mu)) * P_1^2$:

$$|a_2|^2 \leq \frac{|\tau+\mu|}{|\Psi_3 - \frac{\mu\Psi_2^2}{\tau+\mu}|}$$

$$|a_2| \leq \sqrt{\frac{|\tau+\mu|}{|\Psi_3 - \frac{\mu\Psi_2^2}{\tau+\mu}|}}$$

Therefore, the coefficient bound for a_2

$$|a_2| \leq \min\left\{\frac{|\tau-\mu|}{|\Psi_2|}, \sqrt{\frac{|\tau-\mu|}{|\Psi_3 - \frac{\mu\Psi_2^2}{\tau-\mu}|}}\right\}$$

For a_3 $$2\Psi_3 a_3 - 2\Psi_3 a_2^2 = P_1(c_2 - d_2)$$

$$a_3 = a_2^2 + \frac{P_1}{2\Psi_3}(c_2 - d_2)$$

$$|a_3| \leq |a_2|^2 + \frac{|P_1|}{|\Psi_3|}$$

$$|a_3| \leq \max\left\{\frac{|P_1|}{|\Psi_3|}, \frac{|P_2|}{|\Psi_3|}\right\}$$

4. Fekete–Szegö Inequality

Theorem 4.1. Let f $\in\Sigma_B(\mu, p, q)$. Then for a complex parameter η ∈ ℂ, the following Fekete–Szegö inequality holds:

$$|a_3 - \eta a_2^2| \leq \frac{|\tau-\mu|}{|\Psi_3|} \max\left\{1, \left|\frac{P_2}{P_1} - \eta\frac{\Psi_3}{\Psi_2^2}\right|\right\}$$

5. Conclusion

In this work, we have introduced a new subclass of bi-univalent functions associated with balancing polynomials and established coefficient bounds and Fekete–Szegö inequalities. The methods employed combine (p,q)-calculus with polynomial subordination, leading to results that generalize several existing ones. Future work may involve extending these investigations to higher-order Hankel determinants and exploring analogous subclasses defined via other polynomial families.

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