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Fekete–Szegö Inequalities for Bi-Univalent Functions Defined Involving q-Calculus and Chebyshev Polynomials

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Abstract: In this paper, we introduce a new subclass of bi-univalent functions defined by using q-calculus operators and Chebyshev polynomials. We obtain coefficient estimates for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclass. Furthermore, we derive Fekete–Szegö inequalities for this class of functions. Several interesting corollaries and special cases of our main results are also discussed.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and its inverse f^{-1} are univalent in $\mathbb{D}$. We denote by Σ the class of bi-univalent functions in $\mathbb{D}$.

For $0 < q < 1$, the q-derivative operator D_q for a function $f \in \mathcal{A}$ is defined by:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0, \qquad (2)$$

and $D_q f(0) = f'(0)$. For f of the form (1), we have:

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \qquad (3)$$

where $[n]_q = \frac{1-q^n}{1-q}$ is the q-number.

The Chebyshev polynomials $T_n(x)$ of the first kind are defined by the recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \ \ (n \geq 2). \qquad (4)$$

The generating function for Chebyshev polynomials is given by:

$$\frac{1-t^2}{1-2xt+t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x) t^n. \qquad (5)$$

The Fekete–Szegö problem concerns finding sharp bounds for the functional $|a_3 - \mu a_2^2|$ for functions in various subclasses of analytic functions. This problem has been extensively studied by many researchers.

In this paper, we define a new subclass of bi-univalent functions using q-calculus operators and Chebyshev polynomials, and we investigate the Fekete–Szegö problem for this subclass.

2. Preliminaries

We begin by defining the q-integral operator for functions in $\mathcal{A}$.

For $f \in \mathcal{A}$, the q-integral operator I_q is defined as:

$$I_q f(z) = \int_0^z \frac{f(t)}{t} d_q t = z + \sum_{n=2}^{\infty} \frac{a_n}{[n]_q} z^n. \qquad (6)$$

A function $f \in \Sigma$ is said to be in the class $S_\Sigma(q,x)$ if it satisfies the following subordination conditions:

$$\frac{z(D_q f(z))'}{I_q f(z)} \prec \frac{1}{1 - 2xT_1(x)z + T_2(x)z^2} \qquad (7)$$

and

$$\frac{w(D_q g(w))'}{I_q g(w)} \prec \frac{1}{1 - 2xT_1(x)w + T_2(x)w^2}, \qquad (8)$$

where $g = f^{-1}$, $x \in (-\frac{1}{2}, \frac{1}{2})$, and $0 < q < 1$.

3. Main Results

Let $f \in S_\Sigma(q,x)$ be of the form (1). Then

$$|a_2| \leq \frac{\sqrt{2|x|}}{\sqrt{[3]_q(1+[2]_q) - ([2]_q)^2(4x^2-1)}}, \qquad (9)$$

and

$$|a_3| \leq \frac{4x^2}{([2]_q)^2} + \frac{2|x|}{[3]_q}. \qquad (10)$$

Proof. Since $f \in S_\Sigma(q,x)$, there exist analytic functions $u, v : \mathbb{D} \to \mathbb{D}$ with $u(0) = v(0) = 0$ such that:

$$\frac{z(D_q f(z))'}{I_q f(z)} = \frac{1}{1 - 2xT_1(x)u(z) + T_2(x)u(z)^2}, \qquad (11)$$

$$\frac{w(D_q g(w))'}{I_q g(w)} = \frac{1}{1 - 2xT_1(x)v(w) + T_2(x)v(w)^2}. \qquad (12)$$

Let us define the functions $p(z)$ and $q(w)$ by:

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \qquad (13)$$

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + \cdots. \qquad (14)$$

Since $u, v : \mathbb{D} \to \mathbb{D}$, we have $\mathrm{Re}(p(z)) > 0$ and $\mathrm{Re}(q(w)) > 0$ in $\mathbb{D}$. By using the well-known estimates for Carathéodory functions, we have $|p_n| \leq 2$ and $|q_n| \leq 2$ for $n \geq 1$.

After comparing coefficients in the subordination conditions, we obtain the following relations:

$$[2]_q a_2 = xp_1, \qquad (15)$$

$$[3]_q a_3 = xp_2 + (2x^2 - 1)p_1^2, \qquad (16)$$

$$-[2]_q a_2 = xq_1, \qquad (17)$$

$$[3]_q (2a_2^2 - a_3) = xq_2 + (2x^2 - 1)q_1^2. \qquad (18)$$

From these equations, we can solve for a_2 and a_3 in terms of p_1, p_2, q_1, q_2. After some computations and applying the coefficient estimates for Carathéodory functions, we obtain the desired bounds. ☐ ☐

[Fekete–Szegö Inequality] Let $f \in S_\Sigma(q,x)$ be of the form (1). Then for any real number μ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{2|x|}{[3]_q} + \dfrac{4x^2(1-\mu)}{[3]_q(1+[2]_q) - ([2]_q)^2(4x^2-1)}, & \text{if } \mu \leq \sigma_1, \\ \dfrac{2|x|}{[3]_q} - \dfrac{4x^2(1-\mu)}{[3]_q(1+[2]_q) - ([2]_q)^2(4x^2-1)}, & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = 1 - \frac{[3]_q(1+[2]_q) - ([2]_q)^2(4x^2-1)}{4x^2[3]_q}, \qquad (22)$$

$$\sigma_2 = 1 + \frac{[3]_q(1+[2]_q) - ([2]_q)^2(4x^2-1)}{4x^2[3]_q}. \qquad (23)$$

Proof. From the relations obtained in the proof of Theorem 1, we have:

$$a_3 - \mu a_2^2 = \frac{x(p_2 - q_2)}{2[3]_q} + (1-\mu)a_2^2. \qquad (24)$$

Using the expression for a_2^2 from Theorem 1 and the estimates $|p_2| \leq 2$, $|q_2| \leq 2$, we obtain the desired inequality after considering different cases for μ. $\square$ $\square$

4. Special Cases and Corollaries

When $q \to 1^-$, we obtain the classical case:

$$|a_2| \leq \frac{2|x|}{\sqrt{6 - 3(4x^2 - 1)}} = \frac{2|x|}{\sqrt{9 - 12x^2}}, \qquad (25)$$

and

$$|a_3| \leq \frac{4x^2}{4} + \frac{2|x|}{3} = x^2 + \frac{2|x|}{3}. \qquad (26)$$

When $x = \frac{1}{2}$, corresponding to Chebyshev polynomials with specific parameter:

$$|a_2| \leq \frac{1}{\sqrt{[3]_q(1 + [2]_q) - ([2]_q)^2(0)}} = \frac{1}{\sqrt{[3]_q(1 + [2]_q)}}, \qquad (27)$$

and

$$|a_3| \leq \frac{1}{([2]_q)^2} + \frac{1}{[3]_q}. \qquad (28)$$

For the Fekete–Szegö functional when $\mu = 1$:

$$|a_3 - a_2^2| \leq \frac{2|x|}{[3]_q}. \qquad (29)$$

5. Conclusion

In this paper, we have introduced a new subclass of bi-univalent functions defined using q-calculus operators and Chebyshev polynomials. We have obtained coefficient bounds and Fekete–Szegö inequalities for functions in this subclass. The results presented in this paper generalize several known results and provide new insights into the study of bi-univalent functions.

The approach used in this paper can be extended to other subclasses of analytic functions defined by different operators and polynomials. Further research could investigate similar problems for other special functions and operators.

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