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NEIGHBOURHOOD DEGREE-BASED TOPOLOGICAL INDICES OF LADDER TYPE GRAPHS

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1. INTRODUCTION:

Neighbourhood degree-based topological indices have emerged as an important class of molecular descriptors in graph theory due to their ability to incorporate not only the degree of a vertex but also the degree information of its adjacent vertices. In comparison with conventional degree-based indices, they provide a more refined representation of the structural features of molecular graphs and have shown notable effectiveness in QSPR and QSAR modeling. Owing to these desirable properties, several neighbourhood-based descriptors, including the neighbourhood Zagreb index, neighbourhood forgotten index, modified neighbourhood forgotten index, neighbourhood hyper-Zagreb index, and neighbourhood second Zagreb index, have been introduced and studied for a variety of graph families.

The neighbourhood degree sum of a vertex v, denoted δ_v, is defined as the sum of the degrees of all neighbours of v:

$$\delta_v = \sum_{u \in N(v)} d_u$$

Definition 1. Neighbourhood Zagreb index is defined as:

$$M_N(G) = \sum_{v \in V(G)} \delta_G(v)^2$$

Definition 2. Neighbourhood Forgotten index is defined as:

$$F_N(G) = \sum_{v \in V(G)} \delta_G(v)^3$$

Definition 3. Modified neighbourhood forgotten index is defined as:

$$F_N*(G) = \sum_{uv \in V(G)} [\delta_G(u)^2] + [\delta_G(v)^2]$$

Definition 4. Neighbourhood hyper-Zagreb index is defined as:

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$$HM_N(G) = \sum_{uv \in V(G)} [\delta_G(u) + [\delta_G(v)]^2$$

Definition 5. Neighbourhood Second Zagreb index is defined as:

$$M_2*(G) = \sum_{uv \in V(G)} [\delta_G(u)[\delta_G(v)]$$

Ladder graphs and their related variants have received extensive attention in chemical graph theory because of their structural similarity to polymer chains, nanotubular systems, and other molecular architectures. More recently, neighbourhood degree-based descriptors have been investigated for product graphs due to their greater sensitivity to local structural characteristics.

The objective of this chapter is to obtain closed-form expressions for five neighbourhood degree-based topological indices of ladder-type graphs. The analysis begins with the standard ladder graph, defined as the Cartesian product $P_n \square P_2$, and is then extended to its variants, including the open ladder graph and the sliding ladder graph. Furthermore, ladder-type graphs constructed over the path, cycle, and star graph families are investigated. The results are derived using a systematic edge-partitioning approach based on neighbourhood degree sums, and a number of theorems are established for the respective graph families.

2. LITERATURE REVIEW:

A parallel and complementary line of research has focused on neighbourhood degree-based topological indices, in which the descriptor of a vertex is computed not from its own degree but from the sum of the degrees of its neighbours the neighbourhood degree sum. This concept, first proposed by Mondal, De and Pal (2019) in two companion papers introducing a general family of neighbourhood degree-based indices, was motivated by the observation that neighbourhood-based descriptors capture second-order structural information that ordinary degree-based indices cannot. Mondal, De and Pal (2019) demonstrated the practical value of this approach by applying their neighbourhood indices to finite graphene lattices, showing through QSPR analysis that these descriptors correlate strongly with selected physicochemical properties of graphene-like structures. In a related study, Mondal, De and Pal (2020) introduced four additional neighbourhood degree-based indices and, using the benchmark octane-isomer dataset, showed that these indices correlate well with physicochemical properties and exhibit good isomer-discrimination behaviour compared with classical descriptors.

Building on this foundation, Mondal, De and Pal (2021) defined the neighbourhood Zagreb index specifically, derived explicit closed-form expressions for several graph products

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including Cartesian, tensor and wreath products and demonstrated that it predicts physicochemical properties of molecular and nanostructure graphs with accuracy competitive with, and in several cases superior to, classical Zagreb-type indices. Mondal, Dey, De and Pal (2021) extended this work further by proposing six novel neighbourhood-degree-sum-based descriptors (denoted ND_1 through ND_6 and related indices) and conducting a comprehensive QSPR analysis, reporting particularly strong predictive power for the acentric factor of octane isomers. These foundational papers by Mondal and collaborators have since been cited extensively as the theoretical basis for neighbourhood-index research across both nanostructure and pharmaceutical applications.

Subsequent studies extended neighbourhood-index methodology to additional graph families and molecular systems. Ravi and Desikan (2021) computed neighbourhood degree-based indices for graphene structures, while Ravi, Grace and Desikan (2023) extended the open- and closed-neighbourhood-degree-sum framework to metal–insulator transition superlattice networks. Sun, Khalid, Usman, Ahmad, Siddiqui and Fufa (2022) performed a neighbourhood-degree-based topological analysis of the polyphenylene network, deriving closed-form expressions for several neighbourhood indices relevant to polymer chemistry. Chamua, Buragohain and Bharali (2024) conducted a systematic comparison of neighbourhood-degree-sum-based versus classical degree-based indices in a QSPR analysis of alkanes ranging from n-butane to nonane, concluding that the neighbourhood-based descriptors offered improved correlation for several physicochemical endpoints. Shankar et al., (2022) studied multiplicative versions of several neighbourhood degree–based topological indices that had been recently introduced in the literature. They derived closed formulas for these multiplicative neighbourhood indices on certain classes of graphs and analyzed their mathematical properties. Using QSPR methods, they tested how well these indices correlate with physicochemical properties of polycyclic aromatic compounds, showing strong predictive performance.

Afridi et al. (2022) introduced generalized versions of the harmonic, geometric–arithmetic, Kulli–Basava and power-sum-connectivity indices, computing them for bridge graphs over path, cycle and complete graphs as well as for wheel, gear, helm and square-lattice graphs, and correlated these generalized indices with the physicochemical properties of alkane isomers, obtaining values close to experimental observations. Basavanagoud, Sunilkumar and Anand (2018) had earlier introduced the first neighbourhood Zagreb index specifically for several

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classes of nanostructures, providing one of the earliest extensions of neighbourhood-index theory beyond simple hydrocarbon graphs.

Regarding standard and ladder-type graph families specifically the structural class most directly relevant to this thesis, Chatterjee et al. (2024) studied several derived ladder graphs, including slanting, diagonal and open-diagonal ladders, computing their M-polynomials and extracting from them closed-form expressions for Zagreb-, Randić- and other degree-based indices. Their work explicitly highlighted the chemical relevance of ladder-like structures for modelling conjugated systems and polymer backbones, providing direct motivation for the extension of neighbourhood degree-based indices to ladder-type graph families investigated in Chapter 3 of this thesis. Despite the breadth of this literature, a systematic derivation of neighbourhood degree-based indices for ladder-type graphs constructed specifically over standard graph families — path, cycle and star graphs, as pursued in this thesis — has not previously been reported.

3. LADDER-TYPE GRAPHS VIA CARTESIAN PRODUCT

Since the ladder graph is obtained through the Cartesian product operation, we briefly recall its definition. Given two graphs α and β, their Cartesian product $\alpha \square \beta$ is defined as the graph with vertex set $V(\alpha) \times V(\beta)$, where an ordered pair (u_1,u_2) is adjacent to (v_1,v_2) under exactly one of the following two conditions: $u_1=v_1$ and $u_2v_2 \in E(\beta)$ or $u_2 = v_2$ and $u_1v_1 \in E(\alpha)$**.** This operation is employed throughout this chapter to generate ladder graphs and their variants.

3.1 Ladder Graph $L_n=P_n \square P_2$

The ladder graph L_n is defined as the Cartesian product of the path graph P_n and P_2. It is a simple, connected graph with $2n$ vertices and $3n-2$ edges. Its structural uniformity makes the ladder graph a useful and significant class for the study of degree-based topological indices, as its edge distribution can be systematically analyzed through degree partitioning.

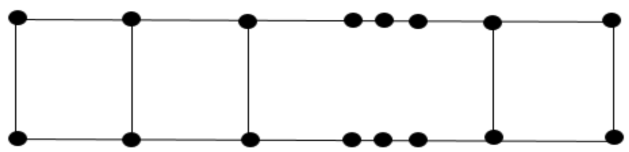

Fig. 3.1 Ladder graph

Theorem 3.1: Let G be ladder Graph L_n, $n \geq 5$, their neighbourhood indices are

$$M_N(G) = 356 + 162(n-4)$$

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$$F_N(G) = 2548 + 1458(n-4)$$

$$F_N^*(G) = 1292 + 162(3n-14)$$

$$HM_N(G) = 2544 + 324(3n-14)$$

$$M_2^*(G) = 626 + 81(3n-14)$$

Proof:

The neighbourhood degree of each vertex in the ladder graph is determined from the degrees of its adjacent vertices refer Fig 3.1. The four corner vertices have degree 2 and are adjacent to one vertex of degree 2 and other vertex of degree 3, yielding a neighbourhood degree of 5 i.e., δ(3) =5. The other four vertices adjacent to corner vertices their neighbourhood vertices δ(3) =2+3+3=8. The remaining 2(n−4) deep internal vertices have degree 3, each adjacent to three vertices of degree 3, resulting in a neighbourhood degree of 9 i.e., δ(3) =9. Total = 4 + 4 + 2(n−4) = 2n, which matches the ladder's total vertex count 2n.

These neighbourhood degree values are subsequently used to partition the edge set to compute the neighbourhood topological indices. Now we already have the neighbourhood degree of every vertex and the neighbourhood edge partition; classify edges by the pair of neighbourhood sums (δ(u),δ(v)) presented in Table 3.1.

Table 3.1 Neighbourhood degree edge partition for ladder graph

Neighbourhood degrees – Edge partition (δ(u),δ(v))	No. of edges
5,5	2
5,8	4
8,8	2
8,9	4
9,9	3n-14

Since total edges = 2+4+2+4+(3n−14) = 3n−2, which matches the ladder's known edge count.

By substituting value from Table 3.1 in the Definition 1-5

$$M_N(G) = \sum_{v\in V(G)} \delta_G(v)^2 = 4.5^2 + 4.8^2 + 2(n-4).9^2$$

$$= 356 + 162(n-4)$$

$$F_N(G) = \sum_{v\in V(G)} \delta_G(v)^3 = 4.5^3 + 4.8^3 + 2(n-4).9^3$$

$$= 2548 + 1458(n-4)$$

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$$F_N*(G) = \sum_{uv\in V(G)}[\delta_G(u)^2]+[\delta_G(v)^2] = 2(5^2+5^2) +4(5^2+8^2)+2(8^2+8^2)+4(8^2+9^2)+(3n-14)(9^2+9^2)$$

$$= 1292 + 162(3n-14)$$

$$HM_N(G) = \sum_{uv\in V(G)}[\delta_G(u)+[\delta_G(v)]^2$$

$$= 2(5+5)^2+4(5+8)^2+2(8+8)^2 + 4(8+9)^2+ (3n-14)(9+9)^2$$

$$= 2544 +324(3n-14)$$

$$M_2*(G) = \sum_{uv\in V(G)}[\delta_G(u)[\delta_G(v)] = 2(5.5)+4(5.8)+2(8.8)+ 4(8.9)+(3n-14)(9.9)$$

$$= 626 + 81(3n-14)$$

Hence, the theorem is proved.

Corolary 3.1:

For n=5, L_5 has 10 vertices and 13 edges. Direct computation of δ (v) for every vertex gives three classes:

Table 3.2 Neighbourhood degree distribution of L_5

δ(u), δ(v)	No. of edges
5,5	2
5,8	4
8,8	2
8,9	4
9,9	1

From Table3.2, $M_N(G)$=518, $F_N(G)$=4006, $F_N(G)*$=1454, $HM_N(G)$=2868, $M_2*(G)$=707.

3.2 Open ladder graph

The open ladder graph OL_n is obtained from the standard ladder graph $L_n=P_n \square P_2$ by removing the two rungs at both ends, deleting the first and last horizontal edges connecting the two path rows. This modification leaves the two terminal vertices on each side connected only within their respective path row, effectively "opening" both ends of the ladder.

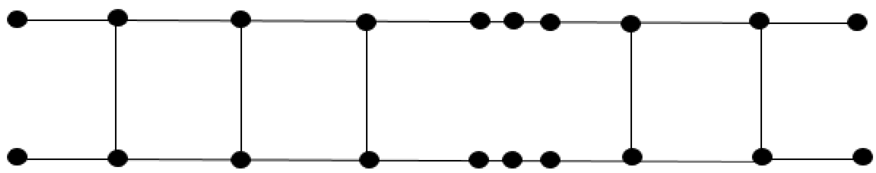

Fig. 3.2 Open ladder graph

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Theorem 3.2: Let G be Sliding ladder Graph OL_n, $n \geq 5$, their neighbourhood indices are

$$M_N(G) = 232 + 162(n-4)$$

$$F_N(G) = 1480 + 1458(n-4)$$

$$F_N^*(G) = 948 + 162(3n-14)$$

$$HM_N(G) = 1816 + 324(3n-14)$$

$$M_2^*(G) = 434 + 81(3n-14)$$

Proof:

The neighbourhood degree of each vertex in the Open ladder graph is obtained by considering the degrees of its adjacent vertices, as illustrated in Fig. 3.2. The four corner vertices, each of degree 1, are adjacent to one vertex of degree 3 therefore, each has neighbourhood degree 3, that is, $\delta(1)=3$ and the other four vertices with degree 3 adjacent to corner vertices, their total neighbourhood degree i.e., $\delta(3)=7$. The remaining $2(n-2)$ deep internal vertices have degree 3 and are adjacent to three vertices of degree 3, giving each of them a neighbourhood degree of 9, i.e., $\delta(3)=9$. Totally, $4 + 4 + 2(n-4) = 2n$. These neighbourhood degree values are then used to partition the edge set to compute the corresponding neighbourhood topological indices.

Since the neighbourhood degree of every vertex has now been determined, the edges can be classified according to the ordered pairs of neighbourhood sums $(\delta(u),\delta(v))$ as presented in Table 3.3.

Table 3.3 Neighbourhood degree edge partition for Open ladder graph

Neighbourhood degrees – Edge partition (δ(u),δ(v)	No. of edges
3,7	4
7,7	2
7,9	4
9,9	3n-14

Since total edges = $4+2+4+(3n-14) = 3n-4$ which matches L_n's $3n-2$ minus the 2 rungs you removed.

By substituting value from Table 3.3 in the Definition 1-5

$$M_N(G) = \sum_{v \in V(G)} \delta_G(v)^2 = 4.3^2 + 4.7^2 + 2(n-2).9^2$$

$$= 232 + 162(n-4)$$

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$F_N(G) = \sum_{v \in V(G)} \delta_G(v)^3$ = $4.3^3+4.7^3+2(n-2).9^3$

= 1480 + 1458(n-4)

$F_N{}^*(G) = \sum_{uv \in V(G)} [\delta_G(u)^2] + [\delta_G(v)^2]$ = $4(3^2+7^2)+2(7^2+7^2)+4(7^2+9^2)+(3n-14)(9^2+9^2)$

= 948 + 162(3n-14)

$HM_N(G) = \sum_{uv \in V(G)} [\delta_G(u) + [\delta_G(v)]^2$ = $4(3+7)^2+2(7+7)^2+4(7+9)^2+(3n-14)(9+9)^2$

= 1816 + 324(3n-14)

$M_2{}^*(G) = \sum_{uv \in V(G)} [\delta_G(u)[\delta_G(v)]$ = 4(3.7)+2(7.7)+4(7.9)+(3n-14)(9.9)

= 434 + 81(3n-14)

Hence proved.

Corollory 3.2:

For OL_5, n=5: 10 vertices, 11 edges. Their edge partition based on vertex degree.

Table 3.4 Neighbourhood degree distribution of OL_5

δ(u), δ(v)	**No. of edges**
3,7	4
7,7	2
7,9	4
9,9	1

From Table 3.4, $M_N(G)$=394, $F_N(G)$=2938, $F_N(G)^*$=1110, $HM_N(G)$=2140, $M_2{}^*(G)$=515.

3.3 Sliding ladder Graph

The sliding ladder graph SL_n is obtained from L_n by shifting one of the two path rows by one position, so that the rungs of the ladder are no longer perpendicular but diagonal. This modification changes the degree sequence; some internal vertices attain degree 4, producing a richer edge partition than L_n, making the TI computation more involved.

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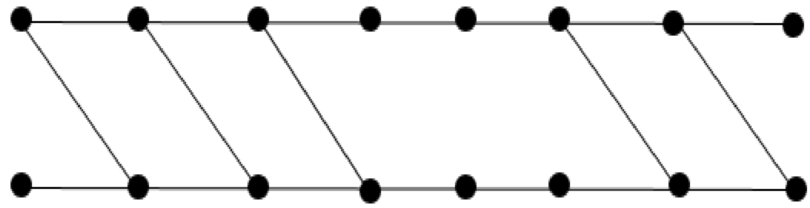

Fig. 3.3 Sliding ladder graph

Theorem 3.3: Let G be Sliding ladder Graph $SL_{n,}$, $n \geq 5$, their neighbourhood indices are

$$M_N\,(G) = 290 + 162(n-4)$$

$$F_N\,(G) = 1942 + 1458(n-4)$$

$$F_N{}^*\,(G) = 1248 + 486(n-5)$$

$$HM_N\,(G) = 2448 + 972(n-5)$$

$$M_2{}^*\,(G) = 600 + 243(n-5)$$

Proof:

The neighbourhood degree of each vertex in the Sliding ladder graph is computed from the degrees of its adjacent vertices, as shown in Fig. 3.3. The two corner vertices, each having degree 1, are adjacent to one vertex of degree 3 hence, their neighbourhood degree is 3, i.e., $\delta(1)=3$, vertices with degree two adjacent with degree one, its neighbourhood degrees are 6 i.e.,.$\delta(2)=6$. The remaining vertices with degree 3 adjacent to degree 2 have $\delta(3)=8$. Finally, $2(n-4)$ internal vertices each have degree 3 and are adjacent to three vertices of degree 3, so their neighbourhood degree is 9, i.e., $\delta(3)=9$. Totally, $2+4+2+2(n-4) = 2n$. These neighbourhood degree values are then employed to partition the edge set for evaluating the neighbourhood topological indices.

With the neighbourhood degree of every vertex determined, the edges are classified according to the pairs $(\delta(u),\delta(v))$, as listed in Table 3.5.

Table 3.5 Neighbourhood degree edge partition for Sliding ladder graph

Neighbourhood degrees – Edge partition (δ(u),δ(v))	**No. of edges**
3,6	2

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6,6	2
6,8	2
6,9	2
8,9	4
9,9	3(n-5)

Since total edges = 2+2+2+2+4+3(n−5) = 12+3(n−5) = 3n−2.

By substituting the values from Table 3.5 in Definition 1-5

$$M_N(G) = \sum_{v\in V(G)} \delta_G(v)^2 = 2.3^2 + 4.6^2 + 2.8^2 + 2(n-4).9^2$$

$$= 290 + 162(n-4)$$

$$F_N(G) = \sum_{v\in V(G)} \delta_G(v)^3 = 2.3^3 + 4.6^3 + 2.8^3 + 2(n-4).9^3$$

$$= 1942 + 1458(n-4)$$

$$F_N{}^*(G) = \sum_{uv\in V(G)} [\delta_G(u)^2] + [\delta_G(v)^2]$$

$$= 2(3^2+6^2) + 2(6^2+6^2) + 2(6^2+8^2) + 2(6^2+9^2) + 4(8^2+9^2) + (3n-5)(9^2+9^2)$$

$$= 1248 + 486(n-5)$$

$$HM_N(G) = \sum_{uv\in V(G)} [\delta_G(u) + [\delta_G(v)]^2$$

$$= 2(3+6)^2 + 2(6+6)^2 + 2(6+8)^2 + 2(6+9)^2 + 4(8+9)^2 + (3n-5)(9+9)^2$$

$$= 2448 + 972(n-5)$$

$$M_2{}^*(G) = \sum_{uv\in V(G)} [\delta_G(u)[\delta_G(v)] = 2(3.6)+2(6.6)+2(6.8)+2(6.9)+4(8.9)+(3n-5)(9.9)$$

$$= 600 + 243(n-5)$$

Hence, proved the theorem.

Corollary 3.3:

For SL_6, n=6: 12 vertices, 15 edges. Their edge partition based on vertex degree.

Table 3.6 Neighbourhood degree distribution of SL_5

δ(u), δ(v)	**No. of edges**
3,6	2
6,6	2

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6,8	2
6,9	2
8,9	4
9,9	3

From Table 3.6, $M_N(G)$ =614 $F_N(G)$=4858, $F_N(G)^*$=1734, $HM_N(G)$=3420, $M_2^*(G)$=843.

4. NEIGHBOURHOOD TIS OF LADDER-TYPE GRAPHS OVER STANDARD GRAPH FAMILIES

A ladder-type graph over a standard graph family is obtained by replacing each rung of the ladder graph by a path P_n, namely the cycle C_n, or the star S_n. This generalisation extends the ladder concept beyond the linear chain topology and produces a richer family of composite graphs with varying degree distributions and structural properties.

Let m, n ≥2 be integers and let H be a connected graph with two distinguished vertices u and v. Consider two vertex-disjoint copies of the path P_m, with vertex sets $\{u_1,\dots,u_m\}$ and $\{v_1,\dots,v_m\}$ and edges (u_i,u_{i+1}), (v_i,v_{i+1}) for $i=1,\dots,m-1$. For each $i \in \{1,\dots, m\}$, take a copy H_i of H with distinguished vertices u_i',v_i' corresponding to u, v *and* identify u_i' with u_i and v_i' with v_i. The resulting graph is called the ladder-type graph over H and is denoted by L (H; u,v).

It consists of two horizontal paths $u_1,\dots,u_m$, $v_1,\dots,v_m$ and, for each i, a “rung” isomorphic to H joining u_i and v_i; the special case $H = P_n$ yields the ladder-type graph L (P_n; u,v) described in Fig. 3.4.

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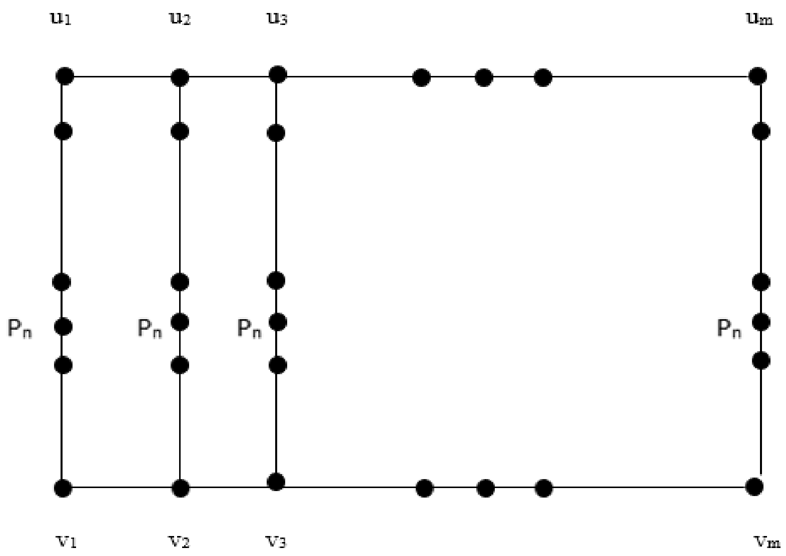

Fig. 3.4 Ladder-type graph over path

Theorem 4.1: Let L ($P_{n;}$ u, v) denote a ladder-type graph over path graph m, n $\geq$ 5and their neighbourhood indices.

$$M_N\,(L\,(P_{n;}\,u, v)\,) = (4+mn-4m)16 + 128(m-4) + 50m + 196$$

$$F_N\,(L\,(P_{n;}\,u, v)\,) = (4 + mn-4m)64 + 1024(m-4) + 250m + 1372$$

$$F_N{}^*\,(L\,(P_{n;}\,u, v)\,) = [2(n-3) +(m-2)(n-5)].32 +178(m-4) +256(m-5) +82m + 1044$$

$$HM_N\,(L\,(P_{n;}\,u, v)\,) = [2(n-3) + (m-2)(n-5).64 + 338(m-4) + 512(m-5) + 162m +2052$$

$$M_2{}^*\,(L\,(P_{n;}\,u, v)\,) = [2(n-3)+(m-2)(n-5)].16 + 80(m-4) + 128(m-5) + 40m + 504$$

Proof:

The neighbourhood degree of each vertex in the ladder-type graph over path graph is computed from the degrees of its adjacent vertices, as shown in Fig. 3.4. The spine end vertices u_1,u_m,v_1,v_m their neighbourhood indices δ(2)=5. Spine deep internal vertices are 2(m-4) have δ(3)=8, Spine near-end vertices u_2,u_{m-1},v_2,v_{m-1} have δ(3)=7. Rung vertices have 4 vertices when i=1, δ(2)=4 when 2 $\leq$ i $\leq$ m-1, they have 2(m-2)vertices, δ(3)=5. Finally, interior rung vertices when 3 $\leq$ j $\leq$ n-2 have m(n-4) vertices δ(2)=4.

Now we have to count the vertices that have distinct δ value.

When δ=4, N_4 = 4+mn-4m

δ=5, N_5 = 2m

δ=7, N_7 = 4

δ=8, N_5 = 2(m-4)

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Table 3.7 Neighborhood degree edge partition for ladder type graph over path graph

Neighbourhood degrees – Edge partition (δ(u),δ(v))	No. of edges
5,7	4
7,8	4
8,8	2(m-5)
4,4	2(n-3)+(m-2)(n-5)
5,4	2m
7,5	4
8,5	2(m-4)

By substituting the values of Table 3.7 in the Definition 1-5

$M_N(G) = \sum_{v\in V(G)} \delta_G(v)^2$ = $N_4.4^2+N_5.5^2+N_7.7^2+N_8.8^2$

= (4+mn-4m) 16 + 128(m-4) + 50m + 196

$F_N(G) = \sum_{v\in V(G)} \delta_G(v)^3$ = $N_4.4^3+N_5.5^3+N_7.7^3+N_8.8^3$

= (4 + mn-4m)64 + 1024(m-4) + 250m + 1372

$F_N*(G) = \sum_{uv\in V(G)} [\delta_G(u)^2]+[\delta_G(v)^2]$

= $|E_{(5,7)}|$ (5^2+7^2) + $|E_{(7,8)}|$ (7^2+8^2) + $|E_{(8,8)}|$ (8^2+8^2) + $|E_{(4,4)}|$ (4^2+4^2) + $|E_{(5,4)}|$ (5^2+4^2) + $|E_{(7,5)}|$ (7^2+5^2) + $|E_{(8,5)}|$ (8^2+5^2)

= [2(n-3) + (m-2)(n-5)].32 +178(m-4) +256(m-5) +82m + 1044

$HM_N(G) = \sum_{uv\in V(G)} [\delta_G(u)+[\delta_G(v)]^2$

= $|E_{(5,7)}|$ $(5+7)^2$ + $|E_{(7,8)}|$ $(7+8)^2$ + $|E_{(8,8)}|$ $(8+8)^2$ + $|E_{(4,4)}|$ $(4+4)^2$ + $|E_{(5,4)}|$ $(5+4)^2$ + $|E_{(7,5)}|$ $(7+5)^2$ + $|E_{(8,5)}|$ $(8+5)^2$

= [2(n-3) + (m-2)(n-5).64 + 338(m-4) + 512(m-5) + 162m +2052

$M_2*(G) = \sum_{uv\in V(G)} [\delta_G(u)[\delta_G(v)]$

= $|E_{(5,7)}|$ (5.7) + $|E_{(7,8)}|$ (7.8)+ $|E_{(8,8)}|$ (8.8) + $|E_{(4,4)}|$ (4.4) + $|E_{(5,4)}|$ (5.4)+ $|E_{(7,5)}|$ (7.5) + $|E_{(8,5)}|$ (8.5)

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$= [2(n-3)+(m-2)(n-5)].16 + 80(m-4) + 128(m-5) + 40m + 504$

Hence proved.

Corollory 4.1 (Consistency relation between the edge-based indices)

For $G = L(P_n; u,v)$ with $m,n\geq5$: $HM_N(G) = F_N^*(G)+2M_2^*(G)$

Proof: For any edge uv, $(\delta(u)+\delta(v))^2 = \delta(u)^2+\delta(v)^2 + 2\delta(u)\delta(v)$. Summing over all edges of G gives the identity directly.

Verification, (m=6, n=6): $F_N^*+2M_2^* = 2468+2(1192) = 4852 = HM_N$

Special case m = n

For $G = L(P_m; u,v)$ with $m=n\geq5$, Theorem 4.1 reduces to:

$$M_N(G)=16m^2+114m-252$$

$$F_N(G)=64m^2+1018m-2468$$

$$F_N^*(G)=32m^2+356m-820$$

$$HM_N(G)=64m^2+692m-1604$$

$$M_2^*(G)=16m^2+168m-392$$

Proof: Substitute n=m into each formula of Theorem 4.1; the bracket term common to F_N^*, HM_N, M_2^* simplifies to $2(m-3)+(m-2)(m-5) = m^2-5m+4$. Collecting powers of m in each expression gives the results above.

Verification (m=n=6): all five match direct computation exactly (1008, 5944, 2468, 4852, 1192).

Example

Ladder-type graph over path $L(P_6; u,v)$, m=6, n=6, 36 vertices, 40 edges. Direct summation gives $M_N(G) =1008$, $F_N(G)=5944$, $F_N(G)^*=2468$, $HM_N(G)=4852$, $M_2^*(G)=1192$.every one of these matches your stated formula exactly.

Theorem 4.2: Let $L(C_n; u, v)$ denote a ladder-type graph over cycle graph $m\geq 5$ and $n \geq 8$ and their neighbourhood indices.

$$M_N (L (C_n; u, v)) = 16m(n-6) + 288(m-4) +144(m-2) +940$$

$$F_N (L (C_n; u, v)) = 64m(n-6) + 3456(m-4) + 864 (m-2) + 8372$$

$$F_N^*(L (C_n; u, v)) = 32m(n-8) + 720(m-4) + 208(m-2) + 576(m-5) + 4096$$

$$HM_N (L (C_n; u, v)) = 64m(n-8) + 1296(m-4) + 400(m-2) + 1152(m-5) + 7872$$

$$M_2^* (L (C_n; u, v)) = 16m(n-8) + 288(m-4) + 96(m-2) + 288(m-5) + 1888$$

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Proof:

The neighbourhood degree of each vertex in the ladder-type graph over a path graph is computed from the degrees of its adjacent vertices, as shown in Fig. 3.5. We now count the vertices with distinct δ values and the neighbourhood edge partition shown in Table 3.8.

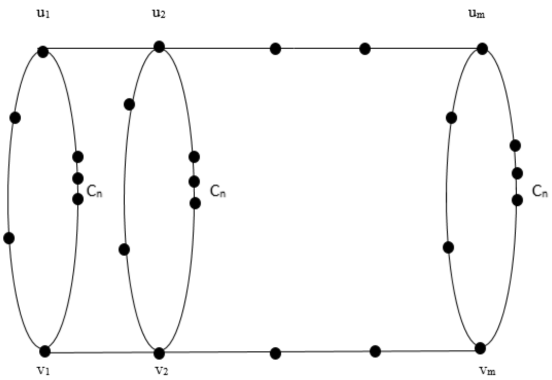

Fig. 3.5 Ladder-type graph over cycle

When $\delta=4$, $N_4 = m(n-6)$

$\delta=5$, $N_5 = 8$

$\delta=6$, $N_6 = 4(m-2)$

$\delta=8$, $N_8 = 4$

$\delta=11$, $N_{11} = 4$

$\delta=12$, $N_{12} = 2(m-4)$

Table 3.8 Neighbourhood degree edge partition for ladder type graph over cycle graph

Neighbourhood degrees – Edge partition (δ(u),δ(v)	No. of edges
8,11	4
11,12	4
12,12	2(m-5)
8,5	8
12,6	4(m-2)
5,4	8

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6,4	4(m-2)
4,4	m(n-8)
6,11	8

By substituting the values of Table 3.8 in the Definition 1-5

$M_N(G) = \sum_{v \in V(G)} \delta_G(v)^2$ = $N_4.4^2+N_5.5^2+N_6.6^2+N_8.8^2+N_{11}.11^2+N_{12}.12^2$

= 16m(n-6) + 288(m-4) +144(m-2) +940

$F_N(G) = \sum_{v \in V(G)} \delta_G(v)^3$ = $N_4.4^3+N_5.5^3+N_6.6^3+N_8.8^3+N_{11}.11^3+N_{12}.12^3$

= 64m(n-6) + 3456(m-4) + 864 (m-2) + 8372

$F_N{}^*(G) = \sum_{uv \in V(G)} [\delta_G(u)^2] + [\delta_G(v)^2]$

= $|E_{(8,11)}|(8^2+11^2) + |E_{(11,12)}|(11^2+12^2) + |E_{(12,12)}|(12^2+12^2) + |E_{(8,5)}|(8^2+5^2) + |E_{(12,6)}|(12^2+6^2) + |E_{(5,4)}|(5^2+4^2) + |E_{(6,4)}|(6^2+4^2) + |E_{(4,4)}|(4^2+4^2) + |E_{(6,11)}|(6^2+11^2)$

= 32m(n−8) + 720(m−4) + 208(m−2) + 576(m−5) + 4096

$HM_N(G) = \sum_{uv \in V(G)} [\delta_G(u) + [\delta_G(v)]^2$

= $|E_{(8,11)}|(8+11)^2 + |E_{(11,12)}|(11+12)^2 + |E_{(12,12)}|(12+12)^2 + |E_{(8,5)}|(8+5)^2 + |E_{(12,6)}|(12+6)^2 + |E_{(5,4)}|(5+4)^2 + |E_{(6,4)}|(6+4)^2 + |E_{(4,4)}|(4+4)^2 + |E_{(6,11)}|(6^2+11^2)$

= 64m(n−8) + 1296(m−4) + 400(m−2) + 1152(m−5) + 7872

$M_2{}^*(G) = \sum_{uv \in V(G)} [\delta_G(u)[\delta_G(v)]$

= $|E_{(8,11)}|(8.11) + |E_{(11,12)}|(11.12) + |E_{(12,12)}|(12.12) + |E_{(8,5)}|(8 \cdot 5) + |E_{(12,6)}|(12.6) + |E_{(5,4)}|(5.4) + |E_{(6,4)}|(6.4) + |E_{(4,4)}|(4.4) + |E_{(6,11)}|(6^2+11^2)$

= 16m(n−8) + 288(m−4) + 96(m−2) + 288(m−5) + 1888

Hence proved the theorem.

Corollary 4.2 (Consistency relation between the edge-based indices)

For G = L(C_n; u,v) with m≥5, n≥8: $HM_N(G)=F_N{}^*(G)+2M_2{}^*(G)$

Proof. For any edge uv, $(\delta(u)+\delta(v))^2 = \delta(u)^2 + \delta(v)^2 + 2\delta(u)\delta(v)$. Summing over all edges of G: i.e. $HM_N(G) = F_N{}^*(G) + 2M_2{}^*(G)$.

Verification, (m=6, n=9): $F_N{}^*+2M_2{}^*$ = 7136 + 2(3232) = 7136+6464 = 13600 = HM_N

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Special case m = n

For G = L(C_m; u,v) with m=n≥8 (i.e. the number of spine copies equals the cycle length), Theorem 3.3.2 reduces to the single-variable polynomials:

$$M_N(G)=16m^2+336m-500$$

$$F_N(G)=64m^2+3936m-7180$$

$$F_N{}^*(G)=32m^2+1248m-2080$$

$$HM_N(G)=64m^2+2336m-3872$$

$$M_2{}^*(G)=16m2+544m-896$$

Proof. Substitute n=m directly into each formula of Theorem 4.2 and collect like powers of m. For instance, for M_N: $M_N(G) = 16m(m-6)+288(m-4)+144(m-2)+940 = 16m^2 + 336m - 500$. The remaining four follow identically.

Verification, (m=n=8): M_N=16(64)+336(8)−500=3212 , F_N=64(64)+3936(8)−7180=28404 , $F_N{}^*$=32(64)+1248(8)−2080=9952, HM_N=64(64)+2336(8)−3872=18912 , $M_2{}^*$=16(64)+544(8)−896=4480, all match direct computation exactly.

Theorem 4.3: Let L($S_{n;}$ u,v) denote a ladder-type graph over a Star graph (each rung: a centre vertex joined to u_i, v_i, and k−2 pendant leaves) m≥5, k≥4, and their neighbourhood indices.

$$M_N\ (L(S_{n;}\ u, v)\ ) = k^3m+k^2m+32km-40k+88m-176$$

$$F_N\ (L(S_{n;}\ u, v)\ ) = k^4m+k^3m+48k^2m-60k^2+264km-528k+496m-1232$$

$$F_N * (L\ (P_{n;}\ u, v)\ ) = 2k^3m+12k^2m-12k^2+88km-144k+216m-492$$

$$HM_N\ (L\ (P_{n;}\ u, v)\ ) = 4k^3m+24k^2m-24k^2+160km-264k+456m-1028$$

$$M_2 * (L\ (P_{n;}\ u, v)\ ) = k^3m+6k^2m-6k^2+36km-60k+120m-268$$

Proof:

The neighbourhood degree of each vertex in the ladder-type graph over a path graph is computed from the degrees of its adjacent vertices, as shown in Fig. 3.6. We now count the vertices with distinct δ values. There are four different vertices; their neighbourhood degree of vertices are presented in Table 3.9.

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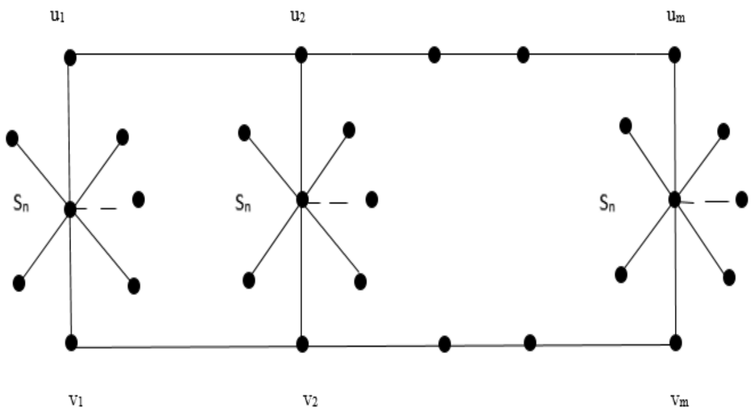

Fig. 3.6 Ladder-type graph over star

When δ=k, N_k = m (k-2)

δ=k+2, N_{k+2} = 2

δ=k+3, N_{k+3} = 4

δ=k+4, N_{k+4} = m-2

δ=k+5, N_{k+5} = 4

δ=k+6, N_{k+6} = 2(m-4)

Table 3.9 Neighbourhood degree edge partition for ladder type graph over star graph

Neighbourhood degrees edge partition	No. of edges
(k, k+1)	2(k-2)
(k, k+4)	(m-2)(k-2)
(k+2, k+3)	4
(k+3, k+5)	4
(k+4, k+5)	4
(k+4, k+6)	2(m-4)
(k+5, k+6)	4
(k+6, k+6)	2(m-5)

By substituting the values of Table 3.9 in Definition 1-5

$$M_N(G) = \sum_{v \in V(G)} \delta_G(v)^2$$

$$= N_k.k^2 + N_{k+2}(k+2)^2 + N_{k+3}(k+3)^2 + N_{k+4}(k+4)^2 + N_{k+5}(k+5)^2 + N_{k+6}(k+6)^2$$

$$= k^3m + k^2m + 32km - 40k + 88m - 176$$

$$F_N(G) = \sum_{v \in V(G)} \delta_G(v)^3$$

$$= N_k.k^3 + N_{k+2}(k+2)^3 + N_{k+3}(k+3)^3 + N_{k+4}(k+4)^3 + N_{k+5}(k+5)^3 + N_{k+6}(k+6)^3$$

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$= k^4m+k^3m+48k^2m-60k^2+264km-528k+496m-1232$

$$F_N*(G)=\sum_{uv\in V(G)}[\delta_G(u)^2]+[\delta_G(v)^2]$$

$= |E_{(k,\,k+2)}|\,(k^2+(k+2)^2) + |E_{(k,\,k+4)}|\,(k^2+(k+4)^2) + |E_{(k+2,\,k+3)}|\,((k+2)^2+(k+3)^2) + |E_{(k+3,\,k+5)}|\,((k+3)^2+(k+5)^2) + |E_{(k+4,\,k+5)}|\,((k+4)^2+(k+5)^2) + |E_{(k+4,\,k+6)}|\,((k+4)^2+(k+6)^2) + |E_{(k+5,\,k+6)}|\,((k+5)^2+(k+6)^2) + |E_{(k+6,\,k+6)}|\,((k+6)^2+(k+6)^2)$

$= 2k^3m+12k^2m-12k^2+88km-144k+216m-492$

$$HM_N(G)=\sum_{uv\in V(G)}[\delta_G(u)+[\delta_G(v)]^2$$

$= |E_{(k,\,k+2)}|\,(k+(k+2))^2 + |E_{(k,\,k+4)}|\,(k+(k+4))^2 + |E_{(k+2,\,k+3)}|\,((k+2)+(k+3))^2 + |E_{(k+3,\,k+5)}|\,((k+3)+(k+5))^2 + |E_{(k+4,\,k+5)}|\,((k+4)+(k+5))^2 + |E_{(k+4,\,k+6)}|\,((k+4)+(k+6))^2 + |E_{(k+5,\,k+6)}|\,((k+5)+(k+6))^2 + |E_{(k+6,\,k+6)}|\,((k+6)+(k+6))^2$

$= 4k^3m+24k^2m-24k^2+160km-264k+456m-1028$

$$M_2*(G)=\sum_{uv\in V(G)}[\delta_G(u)[\delta_G(v)]$$

$= |E_{(k,\,k+2)}|\,(k.(k+2)) + |E_{(k,\,k+4)}|\,(k.(k+4)) + |E_{(k+2,\,k+3)}|\,((k+2).(k+3)) + |E_{(k+3,\,k+5)}|\,((k+3).(k+5)) + |E_{(k+4,\,k+5)}|\,((k+4).(k+5)) + |E_{(k+4,\,k+6)}|\,((k+4).(k+6)) + |E_{(k+5,\,k+6)}|\,((k+5).(k+6)) + |E_{(k+6,\,k+6)}|\,((k+6).(k+6))$

$= k^3m+6k^2m-6k^2+36km-60k+120m-268$

Hence proved the theorem.

Corollary 4.3 (Consistency relation)

For $G=L(S_k;u,v)$ with $m\geq5$, $k\geq4$: $HM_N(G)=F_N^*(G)+2M_2^*(G)$

Verification, $(m=6,k=5)$: $5724+2(2732)=5724+5464=11188=HM_N$

Special case m=k

For $G=L(S_m;u,v)$ with $m=k\geq5$:

$M_N(G)=m^4+m^3+32m^2+48m-176$

$F_N^*(G)=2m^4+12m^3-12m^2-56m-492$

Obtained by substituting k=m into the theorem and collecting powers of m; the remaining three follow the same substitution

Example

For m=6, k=5 (star S_5: 1 centre + 5 leaves, 2 leaves identified with u_i, v_i), L (S_5; u, v) has 36 vertices and 40 edges. Direct computation confirms:

$M_N(G)=2012$, $F_N(G)=17224$, $F_N^*(G)=5724$, $HM_N(G)=11188$, $M_2^*(G)=2732$

matching Theorem 4.3 exactly, and satisfying Corollary 4.3 (5724+2(2732)=11188).

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5. CONCLUSION

This chapter investigated neighbourhood degree-based topological indices for ladder graph L_n, the Sliding Ladder graph SL_n and the Open Ladder graph OL_n constructed via the Cartesian product operation. The study was further extended by considering three standard graph families, path P_n, cycle C_n and star S_n as base graphs in the Cartesian product construction, yielding ladder-type graphs with structurally distinct degree distributions.

Overall, the results confirm that neighbourhood degree-based topological indices are sensitive to both the type of ladder construction and the choice of base graph and that closed-form expressions can be systematically derived for all families using the unified edge partitioning framework developed in this research.

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